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Polynomial Interpolation Methods for Viscous Flow Calculations*

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Higher-order collocation procedures resulting in tridiagonal matrix systems are derived from polynomial spline interpolation and by Hermitian (Taylor series) finite-difference discretization. The similarities and special features of these different developments are discussed. The governing systems apply for both uniform and variable meshes. Hybrid schemes resulting from two different polynomial approximations for the first and second derivatives lead to a nonuniform mesh extension of the so-called compact or Padé difference technique (Hermite 4). A variety of fourth-order methods are described and the Hermitian approach is extended to sixth-order (Hermite 6). The appropriate spline boundary conditions are derived for all procedures. For central finite differences, this leads to a two-point, second-order accurate generalization of the commonly used three-point end-difference formula. Solutions with several spline and Hermite procedures are presented for the boundary layer equations, with and without mass transfer, and for the incompressible viscous flow in a driven cavity. Divergence and nondivergence equations are considered for the cavity. Among the fourth-order techniques, it is shown that spline 4 has the smallest truncation error. The spline 4 procedure generally requires one-quarter the number of mesh points in a given coordinate direction as a central finite-difference calculation of equal accuracy. The Hermite 6 procedure leads to remarkably accurate boundary layer solutions.

1. INTRODUCTION

Three point finite-difference[1] discretization has formed the basis for the overwhelming majority of numerical solutions of the equations of fluid mechanics. For uniform meshes these procedures are typically second-order accurate in the mesh width h. A decrease in order of accuracy results for nonuniform grids. A wide variety of temporal or marching integration schemes have been developed and these include explicit (one step or two step methods) or implicit procedures. For the latter, which generally have better stability properties, the primary advantage of the three-point differencing is that the resulting algebraic matrix system is of a block-tridiagonal[2]

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[1] All future references to finite differences imply second-order accurate central differences.

[2] The blocks are 2×2 for the fourth-order methods and 3×3 for the sixth-order methods.

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form; therefore, an efficient and well developed two-pass algorithm [1] can be applied to invert the matrix operator.

Recently, a number of higher-order numerical methods have been proposed. The obvious extension is to five-point differencing which leads to a fourth-order accurate system. Unfortunately, for implicit integration the matrix system is pentadiagonal and, therefore, the boundaries require special consideration. In addition, the truncation error is considerably larger than that found with the spline and Hermite methods to be discussed. Graves [2] has proposed a five point partial implicit procedure that simplifies the inversion process; although this method is inconsistent in the transient it can be useful for time asymptotic solutions.

A second class of collocation procedures which are also fourth-order accurate for uniform meshes and which retain a 2×2 block-tridiagonal form for the governing matrix system have recently been proposed. These Hermite or spline collocation techniques treat both the functional values and certain derivatives as unknown at the three collocation points. These procedures generally result in a somewhat lower truncation error than that found with a five-point functional discretization and can be derived from appropriate Taylor series expansions (Hermite) or polynomial interpolation (spline). In the former category we have the Padé approximation of Kreiss or so-called compact scheme [3], the Mehrstellung [4] formulation and Hermitian finite-difference developments of Adam [5] and Peters [6]. In the latter group are the spline collocation methods described by Rubin and Graves [7] and Rubin and Khosla [8]. In addition, a spline-on-spline technique is shown to result from a hybrid formulation.

The purpose of the present analysis is (1) to clarify the relationship between the various spline and Hermite developments, (2) to derive the Hermite block-tridiagonal system for a nonuniform mesh, since all previous developments are for uniform meshes,[3] (3) to extend the Hermite philosophy in order to develop a variable mesh sixth-order block-tridiagonal procedure, (4) to briefly review the spline interpolation method, develop this collocation procedure for several new polynomial forms resulting in block-tridiagonal systems, *and to demonstrate that, in fact, all of the results obtained by the Hermite development can be recovered by appropriate spline polynomial interpolation.* Finally, (5) the additional boundary conditions that are required for these higher-order procedures are presented. For finite differences, a second-order accurate two-point boundary condition is derived. A less accurate end-difference formula can be recovered by using extrapolation. The use of polynomial interpolation for higher-order temporal integration is discussed in Ref. [23]. Comparative solutions using second-order accurate finite differences and spline and Hermite formulations are presented for the boundary layer on a flat plate, boundary layers with uniform and variable mass transfer, and the viscous incompressible Navier–Stokes equations describing the flow in a driven cavity. Divergence and nondivergence formulations are described for the cavity.

[3] Adam [5] has also derived, independently, the nonuniform mesh extension of the Hermite 4 method. The authors are grateful to one of the reviewers for bringing this reference to our attention. However, the present approach brings out several additional features not presented by Adam.

2. Polynomial Spline Interpolation

Consider a mesh with nodal points x_j such that

$$a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b.$$

Define the mesh width $h_j = x_j - x_{j-1}$, with $\sigma_j = \sigma = h_{j+1}/h_j$. Consider a function $u(x)$ such that at the mesh points x_j, we specify $u(x_j) = u_j$. For the purposes of the present analysis we define the polynomial spline $S(x_j\,; n, k) \equiv S(n, k)$, such that at the mesh points x_j we prescribe $S(x_j\,; n, k) = u_j$. $S(n, k)$ is an nth order polynomial defined on any interval $[j - 1, j]$ and in the set $C^{n-k}[a, b]$; k is defined as the deficiency of the polynomial spline; i.e., we are considering an nth order polynomial having $n - k$ continuous derivatives on $[a, b]$.

The so-called simple spline [1] has deficiency $k = 1$. The familiar cubic spline is a cubic polynomial of deficiency one or $S(3, 1)$. For a more detailed discussion of the properties of polynomial splines see, for example, Ref. [1].

Cubic splines have been widely used for curve fitting and interpolation purposes, but only recently has spline collocation been adapted for the numerical solution of ordinary [9, 10] and partial differential equations [7, 8, 11]. These procedures have been applied to the equations of fluid mechanics by Rubin and Graves [7] and Rubin and Khosla [8]. In these papers the spline collocation technique is described for the basic cubic spline $S(3, 1)$ as well as a higher-order accurate quintic spline of deficiency three $S(5, 3)$. The former has been termed spline 2 and the latter spline 4. In addition, in Ref. [8] it is shown that the basic three-point finite difference discretization formulas can be obtained by considering the quadratic spline of zero deficiency, i.e., $S(2, 0)$.

The general spline interpolation procedure of Refs. [7, 8] can be summarized as follows. An nth order polynomial is specified on the interval $[j - 1, j]$. The $n + 1$ constants are related to the functional values u_{j-1}, u_j, as well as certain spline derivatives m_{j-1}, m_j, M_{j-1}, M_j. m_j, M_j are the spline derivative approximations to the functional derivatives $U'(x_j)$, $u''(x_j)$, respectively. A similar procedure is considered on the interval $[j, j + 1]$. Continuity of derivatives is then specified at x_j. This process results in two coupled equations for m_j, M_j, $j = 1,\ldots, N$. Boundary conditions are required at $j = 0$ and $j = N + 1$. The system is closed by the governing differential equation for $u(x_j)$, where all derivatives are replaced by their spline polynomial approximations m_j, M_j. The details of this procedure for spline 1, 2, 4 are given in Refs. [7, 8] where a variety of explicit, implicit, two step, relaxation and ADI methods are explored.

This spline procedure can be applied to other polynomials of other orders and deficiencies and thereby a variety of systems can be derived. Since the equations of fluid mechanics are second-order we restrict our attention to polynomial splines defined solely by the functional values and the spline first and second derivatives at the nodal points. In addition, only those polynomial splines resulting in at most 3×3 block-tridiagonal matrix systems are considered. In this regard, in addition to splines 1, 2, and 4, i.e., $S(2, 0)$, $S(3, 1)$, $S(5, 3)$, which have previously been described, the

TABLE I

Method	Interpolation polynomial (x_{j-1}, x_j)	Tridiagonal relationship	Other relationships
$S(2, 0)$ Finite difference	$S(x)=u_jt+u_{j-1}(1-t)$	—	$m_j = [u_{j+1}+(\sigma^2-1)u_j-\sigma^2u_{j-1}]/\sigma(1+\sigma)h_j$,
	$+(u_j-u_{j-1}-h_jm_j)t(1-t)$		$M_j = 2\left[u_{j+1}/\sigma-\frac{1+\sigma}{\sigma}u_j+u_{j-1}\right]\Big/[(1+\sigma)h_j^2]$,
$S(3, 1)$ Spline 2	$S(x)=M_{j-1}\frac{h_j^2}{6}(1-t)^3+M_j\frac{h_j^2}{6}t^3$	$M_{j-1}+2(1+\sigma)M_j+\sigma M_{j+1}$	$m_j = \frac{h_j}{3}(M_j+0.5M_{j-1})+\frac{u_j-u_{j-1}}{h_j}$,
	$+\left(u_{j-1}-M_{j-1}\frac{h_j^2}{6}\right)(1-t)$	$=\frac{6}{h_j^2}\left(\frac{u_{j+1}}{\sigma}-\frac{1+\sigma}{\sigma}u_j+u_{j-1}\right)$,	$m_j = -\frac{h_{j+1}}{3}(M_j+0.5M_{j+1})+\frac{u_{j+1}-u_j}{h_{j+1}}$,
	$+(u_j-M_jh_j^2/6)t$	$\sigma m_{j-1}+2(1+\sigma)m_j+m_{j+1}$	$M_j = \frac{2}{h_j}(m_{j-1}+2m_j)-6\frac{u_j-u_{j-1}}{h_j^2}$,
		$=\frac{3}{h_j}\left(\frac{u_{j+1}}{\sigma}+\frac{\sigma^2-1}{\sigma}u_j-\sigma u_{j-1}\right)$,	$M_j = \frac{-2}{h_{j+1}}(m_{j+1}+2m_j)+6\frac{u_{j+1}-u_j}{h_{j+1}^2}$,
$S(5, 3)$ Spline 4	$S(x)=K_{j-1}\frac{h_j^2}{6}(1-t)^3+K_j\frac{h_j^2}{6}t^3$	$K_{j-1}+2(1+\sigma)K_j+\sigma K_{j+1}$	$m_j = \frac{h_j}{3}(K_j+0.5K_{j-1})+\frac{u_j-u_{j-1}}{h_j}$,
	$+\left(u_{j-1}-K_{j-1}\frac{h_j^2}{6}\right)(1-t)$	$=\frac{6}{h_j^2}\left(\frac{u_{j+1}}{\sigma}-\frac{1+\sigma}{\sigma}u_j+{}_ju_{-1}\right)$	$m_j = \frac{-h_{j+1}}{3}(K_j+0.5K_{j+1})+\frac{u_{j+1}-u_j}{h_{j+1}}$,
	$+\left(u_j-K_j\frac{h_j^2}{6}\right)t+G_jt^3(1-t)^2h_j^2/2$	$\sigma m_{j-1}+2(1+\sigma)m_j+m_{j+1}$	$K_j = \frac{2}{h_j}(2m_j+m_{j-1})-6\frac{u_j-u_{j-1}}{h_j^2}$
	$+G_{j-1}t^2(1-t)^3h_j^2/2$	$=\frac{3}{h_j}\left(\frac{u_{j+1}}{\sigma}+\frac{\sigma^2-1}{\sigma}u_j-\sigma u_{j-1}\right)$	$=\frac{-2}{h_j}(2m_j+m_{j+1})+6\frac{u_{j+1}-u_j}{h_{j+1}^2}$,
	where	$\Delta=0$, $S(5,3)\equiv S(3,1)$, $M_j=K_j$	
	$G_j = \frac{\Delta}{6}(K_{j+1}-(1+\sigma)K_j+\sigma K_{j-1})$,		
	$M_j = K_j+G_j$; $\Delta = \frac{1+\sigma^3}{\sigma(1+\sigma)^2}$,		

$S^1(4,0)$	$S(x)=(1-t^3)u_{j-1}+u_jt^3$ $+3(u_j-u_{j-1})t^2(1-t)(2-t)$ $+h_jm_jt^2(2t-3)(1-t)$ $+h_jm_{j-1}t(1-t)^3+\frac{h_j^2}{2}$ $\times M_jt^2(1-t)^2$	$m_{j+1}+(1+\sigma)^2m_j+\sigma^2m_{j-1}$ $=\frac{2}{1+\sigma}\frac{1}{h_j}\left[\frac{1+2\sigma}{\sigma}u_{j+1}\right.$ $\left.+\frac{\sigma-1}{\sigma}(1+\sigma)^3u_j-\sigma^2(2+\sigma)u_{j-1}\right]$	$\frac{\sigma+2}{12}M_j=-\frac{2m_j+m_{j+1}}{3h_{j+1}}$ $+\frac{\sigma^2}{12(1+\sigma)h_{j+1}}\left[\frac{m_{j+1}}{\sigma}+\frac{\sigma^2-1}{\sigma}\right.$ $\left.\times m_j-\sigma m_{j-1}\right]+\frac{u_{j+1}-u_j}{h_{j+1}^2}$ and $\frac{1+2\sigma}{12\sigma}M_j=\frac{2m_j+m_{j-1}}{3h_j}+\frac{1}{12\sigma(1+\sigma)h_j}$ $\times\left[\frac{m_{j+1}}{\sigma}+\frac{\sigma^2-1}{\sigma}m_j-\sigma m_{j-1}\right]-\frac{u_j-u_{j-1}}{h_j^2}$
$S^2(4,0)$	$S(x)=u_{j-1}(1-t)^3+u_jt(3-3t+t^2)$ $-(u_j-u_{j-1})t(1-t)^3$ $-h_jm_jt(1-t)(1+t-t^2)$ $+\frac{h_j^2}{6}M_{j-1}t(1-t)^3+\frac{h_j^2}{6}$ $\times M_jt(1-t)^2(2t+1)$	$\frac{\sigma^2+\sigma-1}{12\sigma}M_{j+1}+\frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma}M_j$ $+\frac{1+\sigma-\sigma^2}{12}M_{j-1}$ $=\frac{1}{h_j^2}\left(\frac{u_{j+1}}{\sigma}-\frac{1+\sigma}{\sigma}u_j+u_{j-1}\right)$	$m_j=\frac{u_j-u_{j-1}}{h_j}+\frac{h_j}{12}$ $\times\left[\frac{1+4\sigma}{\sigma}M_j-\frac{1}{\sigma(1+\sigma)}M_{j+1}\right.$ $\left.+\frac{1+2\sigma}{1+\sigma}M_{j-1}\right]$ and $m_j=\frac{u_{j+1}-u_j}{h_{j+1}}-\frac{h_{j+1}}{12}\left[(4+\sigma)M_j\right.$ $\left.+\frac{2+\sigma}{1+\sigma}M_{j+1}-\frac{\sigma^2}{1+\sigma}M_{j-1}\right]$

Table continued

TABLE I (*continued*)

Method	Interpolation polynomial (x_{j-1}, x_j)	Tridiagonal relationship	Other relationships
$S(5, 1)$	$S(x)=(u_{j-1}+h_j m_j t)(1+3t)(1-t)^3$ $+[u_j-h_j m_j(1-t)]t^3(4-3t)$ $+6\left[u_{j-1}+\frac{h_j^2}{12}M_{j-1}\right]t^2(1-t)^3$ $+6\left[u_j+\frac{h_j^2}{12}M_j\right]t^3(1-t)^2$	$-\frac{1}{2\sigma}M_{j+1}+\frac{3}{2}\frac{1+\sigma}{\sigma}M_j-\frac{1}{2}M_{j-1}$ $=\frac{10}{h_j^2}\left(\frac{u_{j+1}}{\sigma^3}-\frac{1+\sigma^3}{\sigma^3}u_j+u_{j-1}\right)$ $+\frac{1}{h_j}\left(\frac{-4}{\sigma^2}m_{j+1}+6\frac{\sigma^2-1}{\sigma^2}m_j+4m_{j-1}\right)$	—
		and $7m_{j+1}+8(1+\sigma^3)m_j+7\sigma^4 m_{j-1}$ $=\frac{15}{h_j}\left(\frac{u_{j+1}}{\sigma}+\frac{\sigma^4-1}{\sigma}u_j-\sigma^3 u_j\right)$ $+\sigma h_j(M_{j+1}+\frac{3}{2}(\sigma^2-1)M_j-\sigma^2 M_{j-1})$	—
Hermite 1	See Ref. [23]	Same as $S^2(4, 0)$	Same as $S^2(4, 0)$
Hermite 2	see Ref. [23]	Same as $S^1(4, 0)$	Same as $S^1(4, 0)$
Hermite 3	(Spline on spline) (Hybrid)	$m_{j+1}+(1+\sigma)^2 m_j+\sigma^2 m_{j-1}$ $=\frac{2}{1+\sigma}\frac{1}{h_j}\left[\frac{1+2\sigma}{\sigma}u_{j+1}\right.$ $\left.+\frac{\sigma-1}{\sigma}(1+\sigma)^3 u_j-\sigma^2(2+\sigma)u_{j-1}\right]$	—
		and $M_{j+1}+2(1+\sigma)M_j+\sigma M_{j-1}$ $=\frac{3}{h_j}\left[\frac{m_{j+1}}{\sigma}+\frac{\sigma^3-1}{\sigma}m_j-\sigma m_{j-1}\right]$	—

Hermite 4	No single polynomial (Hybrid)	$\frac{\sigma^2+\sigma-1}{12\sigma} M_{j+1} + \frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma} M_j + \frac{1+\sigma-\sigma^2}{12} M_{j-1} = \frac{1}{h_j^2}\left(\frac{u_{j+1}}{\sigma} - \frac{1+\sigma}{\sigma} u_j + u_{j-1}\right)$ and $m_{j+1}+(1+\sigma)^2 m_j+\sigma^2 m_{j-1} = \frac{2}{1+\sigma}\frac{1}{h_j} \times \left[\frac{1+2\sigma}{\sigma} u_{j+1} + \frac{\sigma-1}{\sigma}(1+\sigma)^3 u_j - \sigma^2(2+\sigma)u_{j-1}\right]$	—
Hermite 6	No single polynomial (Hybrid)	$-\frac{\sigma}{18} M_{j-1} + \frac{(1+\sigma)^2}{9\sigma} M_j - \frac{1}{18\sigma} M_{j+1} = \frac{1}{3h_j^2}\left[\frac{3+5\sigma}{\sigma^3(1+\sigma)} u_{j+1} - \left(\frac{3+5\sigma}{\sigma^3(1+\sigma)} + \frac{\sigma(5+3\sigma)}{1+\sigma}\right) u_j + \frac{\sigma(3\sigma+5)}{1+\sigma} u_{j-1}\right] - \frac{1}{h_j}\left[\frac{4+5\sigma}{9\sigma^2(1+\sigma)} m_{j+1} - \frac{(\sigma-1)5(1+\sigma)^2}{9\sigma^2} m_j + \frac{\sigma(5+4\sigma)}{9(1+\sigma)} m_{j-1}\right]$ and $\frac{14+10\sigma-10\sigma^2}{5\sigma} m_{j+1} + \left[\frac{32}{5\sigma} + \frac{2(\sigma-1)}{5\sigma}(8\sigma^4+18\sigma^3+8\sigma^2-2\sigma+8)\right] m_j + \sigma^2 \frac{[14\sigma^2+10\sigma-10]}{5} m_{j-1} = -\frac{6}{h_j}\left[\frac{(\sigma^2-\sigma-1)}{\sigma^2} u_{j+1} - \frac{(\sigma-1)(1+\sigma)^2(1+\sigma^3)}{\sigma^2} u_j + \sigma^2(2^2+\sigma-1)u_{j-1}\right] - \frac{h_j}{5}[(\sigma+1)(\sigma-2)M_{j+1} - 3(\sigma-1)(\sigma+1)^3 M_j + \sigma^2(2\sigma-1)(\sigma+1)M_{j-1}]$	

TABLE II
Truncation Error of Spline Derivatives

	Uniform mesh ($\sigma = 1$)		Nonuniform mesh	
	m_j	M_j	m_j	M_j
$S(2, 0)$	$\frac{h^2}{6}(u''')_j$	$\frac{h^2}{12}(u^{\mathrm{iv}})_j$	$\frac{\sigma h_j^2}{6}(u''')_j$	$\frac{\sigma - 1}{3} h_j(u''')_j + \frac{h_j^2}{12}\frac{1 + \sigma^3}{1 + \sigma}(u^{\mathrm{iv}})_j$
$S(3, 1)$	$\frac{h^4}{180}(u^{\mathrm{v}})_j$	$\frac{h^2}{12}(u^{\mathrm{iv}})_j$	$\sigma\frac{(\sigma - 1)}{24} h_j^3(u^{\mathrm{iv}})_j + \frac{(1 - \sigma + \sigma^2)}{180} h_j^4(u^{\mathrm{v}})_j$	$\frac{1 + \sigma^3}{1 + \sigma}\frac{h_j^2}{12}(u^{\mathrm{iv}})_j$
$S(5, 3)$	$\frac{h^4}{180}(u^{\mathrm{v}})_j$	$\frac{h^4}{360}(u^{\mathrm{vi}})_j$	Same as (3, 1)	$\frac{7h_j^3}{180}(1 + \sigma^2)(\sigma - 1)(u^{\mathrm{v}})_j + h_j^4\left[\frac{\sigma^2}{360} + \frac{(\sigma - 1)^2}{1080}(7\sigma^2 - 2\sigma + 7)\right](u^{\mathrm{vi}})_j$
$S^1(4, 0)$	$\frac{h^4}{180}(u^{\mathrm{v}})_j$	$\frac{h^4}{360}(u^{\mathrm{vi}})_j$	$\frac{\sigma^2}{240}\frac{(1 + \sigma)^2}{1 + \sigma + \sigma^2} h_j^4(u^{\mathrm{v}})^j$	$\sigma(\sigma - 1)\frac{h_j^3}{120}\left(\frac{3 + 5\sigma + 3\sigma^2}{1 + \sigma + \sigma^2}\right)(u^{\mathrm{v}})_j + \left(\frac{\sigma^4 - \sigma^2 + 1}{\sigma^2 + \sigma + 1}\right)\frac{\sigma h_j^4}{120}(u^{\mathrm{vi}})_j$
$S^2(4, 0)$	$\frac{7}{360} h^4(u^{\mathrm{v}})_j$	$\frac{h^4}{240}(u^{\mathrm{vi}})_j$	$\frac{3(\sigma^4 + 1) - 5(1 + \sigma + \sigma^3 + \sigma^4)}{720} h_j^4(u^{\mathrm{v}})_j$	$(\sigma - 1)(2 + 5\sigma + 2\sigma^2)\frac{h_j^3}{180}(u^{\mathrm{v}})_j + [3(\sigma^2 - 1)^2 + \sigma(2\sigma^2 - \sigma + 2)]\frac{h_j^4}{720}(u^{\mathrm{vi}})_j$
$S(5, 1)$	$\frac{h^6}{5040}(u^{\mathrm{vii}})_j$	$\frac{h^4}{720}(u^{\mathrm{vi}})_j$	$\frac{\sigma^2(\sigma - 1)}{15(1 + \sigma^3)}(1 + 6\sigma + 6\sigma^2 + \sigma^3)\frac{h_j^5}{1440}(u^{\mathrm{vi}}) + \frac{\sigma^3}{5040} h_j^6(u^{\mathrm{vii}})_j$	$\frac{\sigma(1 + \sigma^3)}{1 + \sigma}\frac{h_j^4}{720}(u^{\mathrm{vi}})_j$

5-point finite-difference: $\sigma = 1$;

$$u_x = \frac{8(u_{j+1} - u_{j-1}) - (u_{j+2} - u_{j-1})}{12h} + \frac{h^4}{30}(u^{\mathrm{v}})_j,$$

$$u_{xx} = \frac{16(u_{j+1} + u_{j-1} - 2u_j) - (u_{j+2} - 2u_j + u_{j-2})}{12h^2} + \frac{h^4}{90}(u^{\mathrm{vi}})_j,$$

Hermite 6: $\sigma = 1$;

$$m_j = u_x + h^6(u^{\mathrm{vii}})_j/9450; \qquad M_j = u_{xx} + h^6(u^{\mathrm{viii}})_j/66360.$$

equations governing the spline derivatives m_j, M_j for $S(4, 0)$ and $S(5, 1)$ have been evaluated. All of the governing systems for the various procedures are now specified. The spline polynomial on $[j-1, j]$ is also discussed.

1. *Spline formulations.* Consider the polynomial spline, on $[j-1, j]$,

$$S(x; n, k) = \sum_{i=0}^{n} A_i t^i, \qquad t = (x - x_{j-1})/h_j ; \tag{1}$$

at the nodes specify

$$S(x_{j-1} ; n, k) = u_{j-1} ; \qquad S(x_j ; n, k) = u_j . \tag{2}$$

In addition, in order to specify the A_i values, we require some or all of the conditions

$$S'(x_{j-1} ; n, k) = m_{j-1}, \tag{3a}$$

$$S'(x_j ; n, k) = m_j, \tag{3b}$$

$$S''(x_{j-1} ; n, k) = M_{j-1}, \tag{4a}$$

$$S''(x_j ; n, k) = M_j, \tag{4b}$$

where m_j, M_j are the spline derivative approximations of $u'(x)$, $u''(x)$, respectively. The specific relationships (2)–(4) depend on the order and deficiency of polynomial (1). In addition, depending upon the deficiency of a particular spline procedure, the continuity of various derivatives at x_j leads to the necessary relationships between the values of u_l, m_l, and M_l for $l = j-1$, j, and $j+1$. A block tridiagonal system results. The various forms for different polynomials are presented in Table I.

Other polynomial splines can be considered, however, for polynomials of fifth or lower order, the spline formulations presented herein appear to be the most efficient. For higher-order splines, we require that the third- or higher-order spline derivatives be specified in the evaluation of the A_i in (1). These formulations are not discussed here, although the tridiagonal sixth-order accurate system for M_j derivable from $S(6, 0)$ is presented later in this report.

For the polynomial spline formulations presented here, the truncation errors $T(h_j)$ for the various spline derivatives m_j and M_j are depicted in Table II. We recall that

$$m_j = u'(x_j) + T(h_j),$$

$$M_j = u''(x_j) + T(h_j).$$

For completeness the truncation errors $T(h_j)$ are also given for the five-point finite-difference discretization with a uniform grid. Note that although these errors are fourth order, they are somewhat larger than those obtained with any of the fourth-order polynomial spline formulations.

For $\sigma = 1$ the minimum truncation errors of the fourth-order methods are obtained with $S(5, 3)$ and $S^1(4, 0)$. $S^1(4, 0)$ *and* $S^2(4, 0)$ *retain fourth-order accuracy for* m_j *even* *with a variable mesh.* The other fourth-order polynomial splines lead to third-order accurate formulas for m_j with $\sigma \neq 1$. $S(5, 1)$ is sixth order for m_j with $\sigma = 1$ and fifth order with $\sigma \neq 1$. M_j is fourth order in both cases. A sixth-order formulation for M_j (Hermite 6) is given in Table II. From Table II we see that even with $h = 1.0$ there is a significant reduction in truncation error with the higher-order methods. This is due to smaller numerical coefficients in the error terms. More complete details of the derivations of some of the schemes are given in Ref. [23].

3. Taylor Series Formulation—Hermite Collocation

1. *Compact Formulation*

As discussed previously, higher-order finite-difference equations can be derived from Taylor series expansions. For a uniform grid, fourth-order accuracy is achieved with a five-point expansion formula. The resulting system is pentadiagonal with implicit integration procedures. Recently, it has been shown that a compact [3] or Padé approximation transforms the pentadiagonal system for the functional values at the nodes to a 3×3 block-tridiagonal system for the functional values and their derivatives at the nodes.

It has been observed [3] that with

$$m_j = (u_x)_j, \qquad M_j = (u_{xx})_j,$$

$$D^+u_j = (u_{j+1} - u_j)/h_{j+1}, \qquad D^-u_j = (u_j - u_{j-1})/h_j, \tag{5a}$$

$$D^0u_j = 2(u_{j+1}/\sigma - (1 + \sigma)\, u_j/\sigma + u_{j-1})/((1 + \sigma)\, h_j^2), \tag{5b}$$

$$D^*u_j = (u_{j+1}/\sigma + (\sigma^2 - 1)\, u_j/\sigma - \sigma u_{j-1})/((1 + \sigma)\, h_j), \tag{5c}$$

for a uniform mesh, the five-point difference discretization is of the form

$$m_j = (1 - (h^2/6)\, D^+D^-)(u_{j+1} - u_{j-1})/2h,$$
$$M_j = (1 - (h^2/12)\, D^+D^-)(D^+D^-u_j).$$

The truncation errors are given in Table II. These expressions can be rewritten with a Padé or compact approximation such that

$$m_j = \frac{(u_{j+1} - u_{j-1})/2h}{1 + (h^2/6)\, D^+D^-}, \tag{6a}$$

$$M_j = \frac{D^+D^-u_j}{1 + (h^2/12)\, D^+D^-}, \tag{6b}$$

or

$$(1 + (h^2/6)\, D^+D^-)\, m_j = (u_{j+1} - u_{j-1})/2h, \tag{7a}$$

$$(1 + (h^2/12)\, D^+D^-)\, M_j = D^+D^-u_j. \tag{7b}$$

This results in a fourth-order block-tridiagonal interior point system for the function u_j and the derivatives m_j, M_j. As before, the system is closed with the differential equation and appropriate boundary conditions.

System (7) has appeared in a number of places [3] and is termed compact [4], Mehrstellung [5], and Hermitian [6] differencing. System (7) is fourth order with a somewhat smaller truncation error than the five-point difference equations. *Equations (7) have previously been observed in the spline analysis* presented herein (see $S(4, 0)$). Therefore, this compact formulation is the result of two different polynomial spline formulations for m_j, M_j. Derivation (7) does not provide the simpler expressions relating the derivatives m_j, M_j. These expressions are particularly useful in consideration of boundary conditions and in order to eliminate m_j and, thus, reduce the size of the governing matrix system.

2. *Hermitian Collocation*

Alternate derivations of the spline formulations are possible with finite three-point Taylor series expansions; e.g.,

$$u_{j+1} = u_j + h_{j+1}u_x + h_{j+1}^2 u_{xx}/2 + h_{j+1}^3 u_{xxx}/6 + h_{j+1}^4 u_{xxxx}/24, \tag{8a}$$

$$u_{j-1} = u_j - h_j u_x + h_j^2 u_{xx}/2 - h_j^3 u_{xxx}/6 + h_j^4 u_{xxxx}/24, \tag{8b}$$

where

$$m_j = u_x, \qquad M_j = u_{xx}.$$

Using Taylor series expansions for m and M, and depending upon the treatment of higher-order derivatives, a variety of fourth-order schemes can be derived. Adam [15] has recently presented an alternate derivation of Hermite 4. His procedure does not bring out the hybrid character of the compact scheme. The expansion can be further extended to derive a variable grid sixth-order method (see Table I) resulting in a block 3×3 system. The details of these derivations are given in Ref. [23]. These schemes and their relationship with polynomial spline procedures are also included in Table I.

Therefore, it is possible to derive the polynomial spline results of Section 2 with an Hermitian discretization procedure. Moreover, hybrid systems, which represent approximations resulting from multiple spline formulations, can also be conceived. One of these hybrid systems is the variable mesh extension of the so-called Padé or compact differencing scheme. The truncation errors for all possible systems can be obtained from Table II. Finally, the hybrid systems result in a block-tridiagonal form of m_j, M_j. The simpler relations relating m_j directly to M_j found in the polynomial spline formulations are not obtained. This concept has been extended to a sixth-order system in Hermite 6. Higher-order approximations have not been considered.

4. Boundary Conditions

In all of the techniques described in the previous sections the governing system is at most (3×3) block tridiagonal for u_j, m_j, and M_j. Usually one set of boundary conditions for either u_1,[4] m_1, or a linear combination is prescribed. The additional

[4] Similar considerations apply to the other boundary, i.e., u_N, m_N, and M_N, etc.

"spline boundary conditions" are obtained from the governing equations and/or the interpolation polynomial. For spline 2, the additional boundary condition is obtained with the general two-point spline relations given in Table I. A more detailed discussion is given in Ref. [7]. A similar procedure can be considered for spline 4. Some additional modifications are required. These are described in Ref. [8]. For the other spline or Hermite procedures, general two-point formulas do not exist; therefore, an additional relationship between u_1, m_1, and M_1, etc., is required. In this section, these spline conditions are obtained directly from the interpolation polynomials. For spline 2, 4 this procedure would simply lead back to the two-point relationships between u, m, and M.

A review of the boundary conditions follows.

Spline 2. ($\Delta = 0$) In this case $M_j = K_j$. The prescribed boundary conditions along with the equations relating m, M and u, and the governing equation completely determine the required boundary values for u_1, m_1, and M_1. It should be noted that the requirement of the additional boundary conditions is not relaxed even if the governing (3×3) block is reduced to a single tridiagonal system for M_j, as presented in Ref. [7].

Spline 4. In this formulation the (3×3) blocks relate u_j, m_j, and K_j. The boundary conditions on u_1, m_1, and M_1 can be obtained as with spline 2. However, a boundary condition for K_1 is required. A third-order accurate condition for K_1 was obtained in Ref. [8] by using the extrapolation,

$$(u_{xx})_2 - (u_{xx})_1 = M_2 - M_1 = K_2 - K_1 .$$

In order to increase the accuracy of this boundary condition, a fourth-order extrapolation is used here. Therefore, with $\sigma = 1$,

$$M_3 - 2M_2 + M_1 = K_3 - 2K_2 + K_1$$

or

$$K_1 = M_1 - 2G_2 + G_3 = M_1 - h_2^2(u^{\text{iv}})_1/12. \tag{9}$$

For the other formulations a two-point relationship between $(u, m, M)_1$ and $(u, m, M)_2$ can be obtained from the interpolation polynomial. For the finite difference or spline 1 approximation this procedure produces a two-point second-order accurate formula which facilitates the application of derivative boundary conditions.

Spline 1. As shown in Ref. [8],

$$S(x, 2, 0) = u_{j+1}t + u_j(1 - t) - [u_{j+1} - u_j - h_{j+1}m_j]\, t(1 - t) \tag{10}$$

on the interval $(j, j + 1)$. Differentiating twice we obtain for $j = 1$

$$u_2 - u_1 - h_2 m_1 - (h_2^2/2)\, M_1 = 0. \tag{11}$$

Surprisingly this boundary condition has not been used, to the authors' knowledge, for finite-difference calculations. It appears to be a noteworthy improvement on three-point end-difference formulas. The truncation error is $O(h^3)$.

If a lower order expression is used for M_1 in (11), i.e.,

$$M_1 = ((m_2 - m_1)/h_2) + O(h), \tag{12}$$

we obtain

$$u_2 - u_1 - (h_2/2)(m_1 + m_2) = 0. \tag{13}$$

The familiar end-difference formula is recovered with a central-difference approximation for m_2 in (13).

$S^1(x; 4, 0)$. Differentiating the interpolation polynomial, we obtain the results

$$u_j - u_{j-1} - (h_j/3)(2m_j + m_{j-1}) + (h_j^2/6)\, M_j = (h_j^4/72)(u^{\text{iv}})_{j-1} \tag{14}$$

or

$$u_{j+1} - u_j - (h_{j+1}/3)(2m_j + m_{j+1}) - (h_{j+1}^2/6)\, M_j = -h_{j+1}^4/72(u^{\text{iv}})_j . \tag{15}$$

A fourth-order relationship results from (14) if the right-hand side is neglected. This is equivalent to the one used for spline 2, 4. A more accurate relationship can be obtained by using a lower-order extrapolation for $(u^{\text{iv}})_j$ or by eliminating $(u^{\text{iv}})_j$. The latter procedure for $j = 1$ leads to

$$u_2 - u_1 - (h_2/2)(m_2 + m_1) + (h_2^2/12)(M_2 - M_1) = 0. \tag{16}$$

The truncation error in (16) is $O(h^5)$. This expression has previously been obtained by Hirsh [3].

An alternate expression obtained from the polynomial is (17)

$$8(u_j - u_{j-1}) - h_j(5m_j + 3m_{j-1}) + h_j^2 M_j = -(h_j^3/6)(u_{xxx})_{j-1} . \tag{17}$$

Equation (16) results from (17) when a lower-order cubic polynomial is used to relate m_j, M_j, and $(u_{xxx})_j$. This procedure, which relies on lower-order polynomials to approximate higher derivatives, is noteworthy.

$S^2(x; 4, 0)$. The procedure outlined for $S^1(x, 4, 0)$ can also be utilized to derive the required two-point relationship for $S^2(x, 4, 0)$. Equation (15), is recovered. An alternative procedure where both $S^1(x, 4, 0)$ and $S^2(x, 4, 0)$ polynomials are used also leads to Eq. (15). This is similar to the hybrid procedure described previously.

$S(x; 5, 1)$. Three simple two-point relationships which are sixth-order accurate can easily be obtained by differentiating the corresponding interpolation polynomial three, four, and five times, respectively,

$$u_2 - u_1 - \frac{h_2}{5}(2m_2 + 3m_1) + \frac{h_2^2}{20}(M_2 - 3M_1) - \frac{h_2^3}{60}(u_{xxx})_1 = 0, \tag{18a}$$

$$u_2 - u_1 - \frac{h_2}{15}(7m_2 + 8m_1) + \frac{h_2^2}{30}(2M_2 - 3M_1) + \frac{h_2^4}{360}(u^{\text{iv}})_1 = 0, \tag{18b}$$

and

$$u_2 - u_1 - \frac{h_2}{2}(m_2 + m_1) + \frac{h_2^2}{12}(M_2 - M_1) - \frac{h_2^5}{720}(u^{\text{v}})_1 = 0. \quad (18c)$$

In Eqs. (18) the three derivatives $(u_{xxx})_1$, $(u^{\text{iv}})_1$, and $(u^{\text{v}})_1$ can be evaluated by using lower-order extrapolation as before or by differentiating the governing differential equations. All the two-point relationships derived above can also be obtained by the Hermite collocation method.

Hermite 6. Spline boundary conditions are obtained by Hermitian collocation. This leads to a number of two-point formulas. Two of these relations are given below.

$$u_2 - u_1 - \frac{h_2}{5}(2m_2 + 3m_1) + \frac{h_2^2}{20}(M_2 - 3M_1) - \frac{h_2^3}{60}(u_{xxx})_1 - \frac{h_2^6}{1440}(u^{\text{vi}})_1 = 0 \quad (19a)$$

and

$$u_2 - u_1 - \frac{h_2}{2}(m_2 + m_1) + \frac{h_2^2}{12}(M_2 - M_1) - \frac{h_2^5}{720}(u^{\text{v}})_1 - \frac{h_2^6}{180}(u^{\text{vi}})_1 = 0. \quad (19b)$$

The derivatives $(u_{xxx})_1$, $(u^{\text{v}})_1$, and $(u^{\text{vi}})_1$ can be obtained by either lower-order polynomial extrapolation or by differentiating the governing equations.

5. Examples

A. *Similar Boundary Layer: Zero Mass Transfer*

As a first test of the various polynomial spline or Hermite formulations considered in the previous sections, solutions have been obtained for the similarity equations governing laminar boundary layer behavior [12]. This example which has been discussed in previous spline presentations is included here simply to test the applicability of the newly developed Hermite 6 procedure and more accurate fourth-order boundary conditions discussed earlier

$$u'' + fu' + \beta(1 - f'^2) = 0, \qquad u = u(\eta), \qquad f' = u. \quad (20)$$

Primes denote differentiation with respect to η, where $\eta = y(\text{Re}/2x)^{1/2}$; Re is the Reynolds number; y is measured normal to the surface, and x along the surface. The respective velocities are v and u. We approximate the derivatives u_j', u_j'', f' with m_j, M_j and $\overline{m}_j$, respectively, so that the governing Eqs. (20) become

$$\begin{aligned} M_j + f_j m_j + \beta(1 - \overline{m}_j^2) &= 0, \\ \overline{m}_j &= u_j . \end{aligned} \quad (21)$$

The additional equations for m_j, M_j, $\overline{m}_j$ are given in Sections 2 and 3 for each of the polynomial interpolation procedures. The systems are closed with the boundary conditions at the surface $y = 0$ ($j = 1$) and the edge of the boundary layer $y = y_e$ or $j = N$;

$$f_1 = u_1 = 0, \qquad u_N = 1. \tag{22}$$

The additional boundary conditions on m_1, m_N, M_1, M_N are obtained from Eqs. (20), (21), the spline formulas typified by (14)–(16), or from the Hermite expansions (14a) and (14b). The boundary conditions used here have truncation errors that parallel those for the interior systems shown on Table II. For spline 2 and spline 4, boundary conditions have been discussed in greater detail in Refs. [7, 8]; however, only third-order conditions were used for the spline 4 calculations so that the present results are somewhat more accurate.

The results of the polynomial spline calculations are presented in Table III for a variety of uniform and variable meshes. The notation $\sigma = 1.5/1$ means that $h_j = \min\{h_j, 1\}$, and that $\sigma = 1.5$ for $h_j \leqslant 1$. The remarkable accuracy of Hermite 6 with the uniform mesh $h = 1.0$ is noteworthy. It is apparent that significant improvements in accuracy are achieved by considering higher-order polynomial splines.

B. *Similar Boundary Layer*: *Mass Transfer*

In order to carry out more stringent tests of the polynomial methods, boundary layers with surface mass transfer are considered. In this section, similarity solutions corresponding to mass transfer of the type $V_s \sim x^{-1/2}$ are evaluated; in the following section, uniform injection and suction is specified; i.e., $V_s \sim$ constant so that the boundary layer behavior is nonsimilar. The subscript s denotes the surface values. With large injection it is possible to blow the boundary layer off of the surface, and with large suction the boundary layer becomes very thin and the shear stresses

TABLE III

Similar Boundary Layer Solution: $f''(0)$

η_{max}	h_2	σ	Spline 2 $S(3, 1)$	$S^2(4, 0)$	Hermite 4 (compact)	Spline 4 $S(5, 3)$	Hermite 6
				$\beta = 0$[a]			
6.0	0.1	1.0	0.469634	0.469597	0.469600	0.469600	0.469600
20.0	1.0	1.0	0.475357	0.479359	0.473602	0.470730	0.469690
16.078	0.1	1.5/1	0.464325	0.471666	—	0.470025	—
16.063	0.05	1.8/1	0.462008	0.469926	—	0.469438	—
				$\beta = 1$[b]			
6.0	0.1	1.0	1.23227	1.23260	1.23258	1.23259	1.23259
20.0	1.0	1.0	1.20612	1.20863	1.21260	1.21863	1.23242
9.448	.001	1.8/1	1.23604	1.23301	—	1.23299	—

[a] $f''(0) = 0.469600$ (Rosenhead [12]).
[b] $f''(0) = 1.23259$ (Rosenhead [12]).

become quite large. Therefore, these boundary layer profiles are more difficult to approximate numerically, and provide more exacting tests of the spline and Hermite collocation procedures.

The equations governing the similar boundary layer with mass transfer are (20) and (21). The only change is in the boundary conditions (22) for f_1, so that now $f_1 = K$, where $K < 0$ for injection and $K > 0$ for suction.

The results of these calculations are shown in Table IV and Figs. 1 and 2. The figures show velocity profiles for suction and injection, respectively. The flat plate Blasius profile is also included in order to emphasize the extreme thinning of the boundary layer with suction and the blowoff obtained with injection. The polynomial solutions shown in the figures are in excellent agreement with the numerical values of Emmons and Leigh [13]. These profiles are coincident with the polynomial solutions obtained with spline 4 or Hermite 6 and, therefore, are not specifically included in the figures. The second-order accurate finite-difference results are not as accurate and exhibit an erroneous overshoot for the suction case (Fig. 1). For the suction profile only two points lie with the boundary layer. More exact comparisons are shown in Table IV. N_δ denotes the number of grid points within the boundary layer. A variety of results for uniform and nonuniform grids are presented. The polynomial solutions retain a high degree of accuracy for both the high shear suction and near separated injection cases. It is generally found that for equal accuracy, spline 4 requires one-quarter the number of mesh points required in finite-difference calculations; e.g., with $K = 0.5$, $f''(0) = 0.7394$ (81 points with finite difference) and $f''(0) = 0.7392$ (21 points with spline 4). Similar behavior is found with Burgers' equation [23] and the cavity solutions to be discussed later.

TABLE IV

$f''(0)$ Similar Boundary Layer with Nonuniform Mass Transfer

h	σ	K	F.D.	Spline 4	Hermite 6	Ref. [13]	N_δ/N
0.1	1.0	0.5	0.7394	0.7394		0.7394	35/81
0.1	1.5/1.	0.5	0.7842	0.7406			7/21
1.0	1.0	0.5	0.7992	0.7545			3/21
0.1	1.5/1.	10.0	7.8903	6.9817		7.1397	2/21
0.15	1.5/1.	10.0	7.6869	6.8703			1/21
0.3	1.0	10.0	5.2677	7.2178	7.0425		1/21
0.1	1.0	−0.5	0.2326	0.2326		0.2326	48/81
0.1	1.5/1.	−0.5	0.2317	0.2321			9/21
1.0	1.0	−0.5	0.2514	0.2253			5/21
0.1	1.5/1.	−1.2	0.0041	0.0046		0.0047	12/21
1.0	1.0	−1.2	0.0009	0.0045	0.0048		9/21

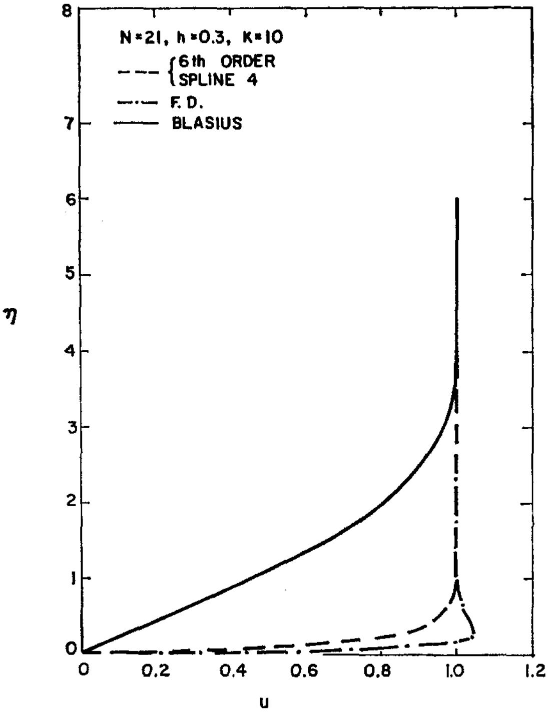


FIG. 1. Similar boundary Layer with Nonuniform suction.

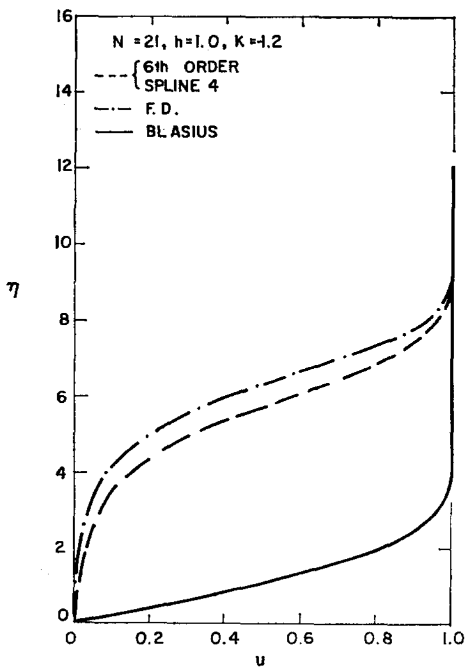


FIG. 2. Similar boundary layer with Nonuniform blowing.

C. Nonsimilar Boundary Layer: Uniform Mass Transfer

With uniform injection or suction V_s = constant, and the following coordinate transformation is applied [12]:

$$\xi = V_s(x\,\mathrm{Re}/2)^{1/2}; \qquad \eta = y(\mathrm{Re}/2x)^{1/2},$$
$$\psi = (2x/\mathrm{Re})^{1/2} f(\xi, \eta); \qquad u = \psi_y = f_\eta,$$
$$v = -\psi_x = (2\mathrm{Re}\, x)^{-(1/2)}(f + \xi f_\xi - \eta f_\eta).$$

The governing boundary layer equations become

$$u_{\eta\eta} + f u_\eta + \xi(f_\xi u_\eta - u u_\xi) = 0 \tag{23a}$$

and the boundary conditions are, at

$$\eta = 0 \quad f = \pm\xi,^{5} \quad u = 0,$$

and

$$\lim \eta \to \infty, \qquad u \to 1. \tag{23b}$$

The spline equations are

$$M_{ij} + (f_{ij} + \xi_i (f_\xi)_{ij})\, m_{ij} = (\xi_i/2)(u_\xi^{\,2})_{ij}$$

where $(f_\xi)_{ij} = (f_{ij} - f_{i-1,j})/\Delta\xi$, and with quasi-linearizatoin

$$(u_\xi^{\,2})_{ij} = (2u_{ij}^* u_{ij} - u_{ij}^{*2} - u_{i-1,j}^2)/\Delta\xi.$$

Iteration is used for the nonlinear term and the asterisk denotes the values from the previous iteration. The equation for $\bar{m}_j$ is the same as given by (21). Once again the spline derivative boundary conditions are obtained from governing Eq. (23) and the derivative relations obtained with the polynomial interpolation procedure given in Section 4.

For $\xi \gg 1$, with suction the classical [12] asymptotic suction profile will be recovered, i.e.,

$$u \sim 1 - \exp(-V_s y\,\mathrm{Re}),$$

or

$$u \sim 1 - \exp(-2\eta\xi). \tag{24}$$

For injection, there has been some question [14] as to whether the boundary layer will separate at a finite ξ location. This question will be addressed in the discussion of results which follows.

The solutions are shown on Tables V and VI and Figs. 3 and 4. With many mesh points all of the methods, including finite difference, work quite well. As the mesh size is increased the finite-difference solutions begin to deviate from the polynomial results.

[5] The positive sign denotes suction and the negative sign denotes injection. V_s is positive in both cases.

TABLE V

$f''(0)$ Nonsimilar Boundary Layer with Uniform Suction

				F.D.		Spline 4		Hermite 4	
ξ	h	σ	N	$u(\xi, \Delta\eta)$	$f''(0, \xi)$	$u(\xi, \Delta\eta)$	$f''(0, \xi)$	$u(\xi, \Delta\eta)$	$f''(0, \xi)$
0.09	1.0	1.0	11	0.5321	0.6798	0.5218	0.5829	0.5228	0.5874
0.49				0.7776	1.0539	0.7357	1.1769	0.7369	1.1860
0.79				0.9185	1.3270	0.8249	1.6775	0.8235	1.7202
0.95				0.9833	1.4580	0.8536	1.9751	0.8491	2.0599
1.0				1.0022	1.4970	0.8600	2.0745	0.8544	2.1777
0.09	0.1	1.5	21	0.0628	0.6340	0.0576	0.5817	0.0577	0.5823
0.49				0.1253	1.3119	0.1119	1.1748	0.1120	1.1762
0.79				0.1761	1.8898	0.1556	1.6822	0.1557	1.6828
0.95				0.2038	2.2151	0.1792	1.9678	0.1792	1.9675
1.0				0.2125	2.3184	0.1866	2.0587	0.1866	2.0581
0.09	0.1	1.0	61	0.0575	0.5807	0.0575	0.5807	0.0575	0.5807
0.49				0.0979	1.0167	0.1122	1.1781	0.1122	1.1780
0.79				0.1566	1.6804	0.1563	1.6902	0.1563	1.6900
0.95				0.1806	1.9629	0.1802	1.9790	0.1817	1.9970
1.0				0.1882	2.0526	0.1877	2.0709	0.1877	2.0707

TABLE VI

$f''(0)$ Nonsimilar Boundary Layer with Uniform Blowing

				F.D.		Hermite 4		Spline 4	
ξ	h	σ	N	$u(\xi, \Delta\eta)$	$f''(0, \xi)$	$u(\xi, \Delta\eta)$	$f''(0, \xi)$	$u(\xi, \Delta\eta)$	$f''(0, \xi)$
0.09	1.0	1.0	31	0.4004	0.4101	0.3859	0.3587	0.3865	0.3618
0.29				0.2429	0.1866	0.2192	0.1619	0.2185	0.1585
0.59				0.0367	−0.0065	0.0128	0.0070	0.0127	0.0069
0.79								1.7×10^{-7}	8.8×10^{-9}
0.09	0.1	1.03	81	0.0364	0.3610	0.0364	0.3607	0.0364	0.3607
0.29				0.0172	0.1672	0.0172	0.1670	0.0172	0.1670
0.59				6.8×10^{-4}	6.3×10^{-3}	6.7×10^{-4}	6.4×10^{-3}	6.7×10^{-4}	6.4×10^{-3}
0.79				4.3×10^{-9}	3.9×10^{-8}	5.5×10^{-9}	5.1×10^{-8}	5.5×10^{-9}	5.1×10^{-8}
0.84				2.7×10^{-11}	2.4×10^{-10}	4.4×10^{-11}	4.0×10^{-10}	4.4×10^{-11}	4.0×10^{-10}

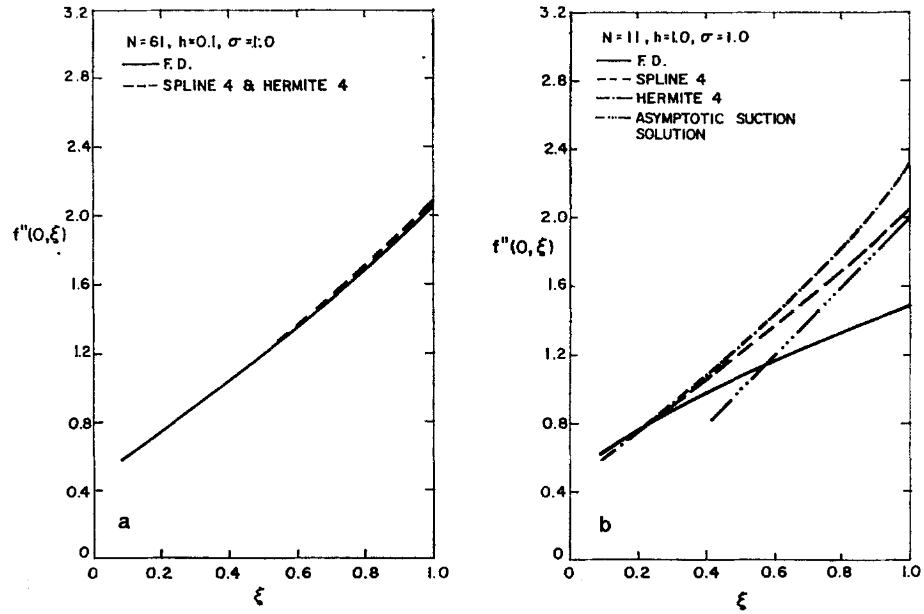


FIG. 3. Shear with uniform suction.

[illegible] suction calculation, the asymptotic solution (24) gives for $\xi \gg 1$

$$f''(0) = 2\xi.$$

Therefore, at $\xi = 1.0$, $f''(0) \approx 2$. The spline 4 results very closely approximate this value; these solutions are in all cases more accurate than the Hermite 4 results. Table V presents the shear values for both the coarse and fine grids. Also shown is the velocity one grid point away from the surface. The asymptotic solution (24) gives at $\xi = 1.0$

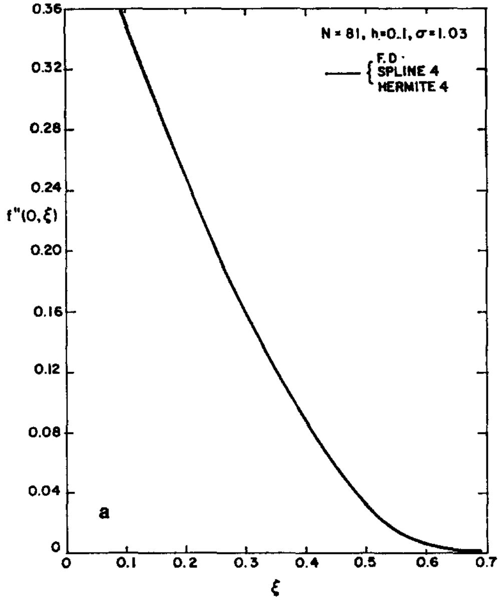
$$u(0.1) = 0.1812$$

or

$$u(1.0) = 0.8647.$$

Once again the spline 4 results are best.

Detailed injection solutions are shown on Table VI. For the very fine grid, there was no indication of separation as inferred in Ref. [14]. This was true for all calculations. The shear decreased but never vanished. For the coarser grid the finite-difference solution did lead to a separation point, but the polynomial solutions still did not separate. This behavior is also depicted on Fig. 4b. The conclusion obtained from these results would appear to be that separation does not occur with uniform injection; instead, the shear decreases asymptotically to zero for large values of ξ.



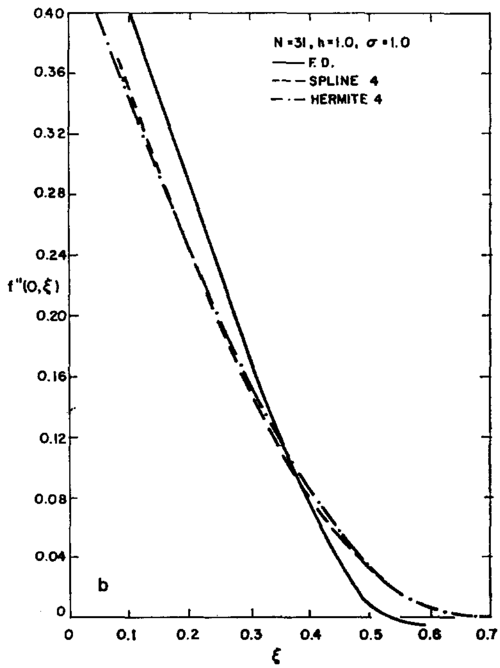


FIG. 4. Shear with uniform blowing.

D. *Incompressible Flow in a Cavity*

D.1. *Nondivergence form.* As a final test problem the laminar incompressible flow in a driven cavity is considered. This problem has been studied extensively by many investigators, see Ref. [15]. The governing equations in terms of a vorticity and stream-function system are

$$\psi_{xx} + \psi_{yy} = \zeta, \tag{25}$$

$$\zeta_t + u\zeta_x + v\zeta_y = (1/\mathrm{Re})(\zeta_{xx} + \zeta_{yy}), \tag{26}$$

where ψ is the stream function, ζ is the vorticity; $u = \psi_y$, and $v = -\psi_x$ are the velocities in the x and y directions, respectively. The boundary conditions and geometry are shown in Fig. 5.

Solutions of (26) are obtained with a predictor-corrector procedure described in Ref. [16]. For Poisson equation (25) a modified version of Buneman's direct solver [17] is used. The spline approximations of (25) and (26) in nondivergence form are

$$L_{ij}^{\psi} + M_{ij}^{\psi} = \zeta_{ij}. \tag{27a}$$

$$\frac{\zeta_{ij}^{n+1} - \zeta_{ij}^{n}}{\Delta t} + u_{ij}(m_{ij}^{\zeta})^{n+1} + v_{ij}(l_{ij}^{\zeta})^{n+1} = \frac{1}{Re}\,[L_{ij}^{\zeta} + M_{ij}^{\zeta}]^{n+1} \tag{27b}$$

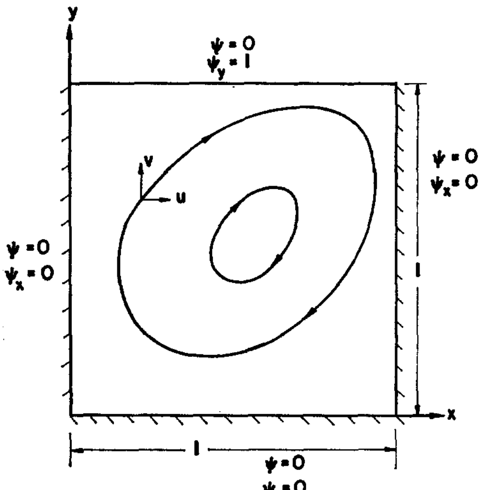


FIG. 5. Schematic of the driven cavity.

where l_{ij} and L_{ij} denote the polynomial approximations of $(\ )_y$ and $(\ )_{yy}$, respectively. The superscripts denote the specific function. The spline boundary conditions are obtained by satisfying (27b) at the walls. The details of this procedure have been previously described in Refs. [4, 7, 8].

Solutions have been obtained for a square cavity with Re = 100. A 17 × 17 point mesh has been considered The results are shown in Table VII and Fig. 6. Table VII compares the Hermite 4 and spline 4 solutions with results obtained in previous studies [7]. The maximum value of the stream function, of the vortex center, and the vorticity at the midpoint of the upper moving wall are presented. It is significant that with a 17 × 17 mesh the spline 4 solutions parallel those obtained with a 57 × 57 point finite-difference discretization. Moreover, the spline 4 results are very close to the extrapolated values as projected for the nondivergence equations. This behavior carries over to the velocity profile shown on Fig. 6.

D.2. *Divergence form.* The governing equations in conservation form are

$$\psi_{xx} + \psi_{yy} = \zeta,$$
$$\zeta_t + (u\zeta)_x + (v\zeta)_y = (1/\text{Re})(\zeta_{xx} + \zeta_{yy}).$$

The spline approximation to these equations becomes,

$$L_{ij}^{\psi} + M_{ij}^{\psi} = \zeta_{ij}$$
$$\frac{\zeta_{ij}^{n+1} - \zeta_{ij}^{n}}{\Delta t} + \tilde{m}_{ij} + \tilde{l}_{ij} = \frac{1}{\text{Re}}(L_{ij}^{\zeta} + M_{ij}^{\zeta}) \tag{27c}$$

where $\tilde{m}_{ij} = (u\zeta)_x$ and $\tilde{l}_{iy} = (u\zeta)_y$.

The method of solution parallels that for the nonconservation equations. Solutions have been obtained for Re = 100 and these results are also included in Table VII and Fig. 6. There is a significant improvement in the results, which is particularly noteworthy, for the coarse meshes. The one to four mesh point correspondence between the spline and finite-difference solutions with equal accuracy is maintained. The 17 × 17 spline 4 solution is remarkably accurate. For the two-dimensional problem considered here, this in effect leads to a sixteen-fold reduction in the number of mesh points. Since the convergence rate is approximately inversely proportional to the square of the number points a significant reduction in computational time is possible. Due to the fact that the spline and finite-difference calculations were made independently and on different computers a meaningful time comparison cannot be included. However the present spline calculations (17 × 17) did require considerably less time and storage than did the (65 × 65) finite-difference results, and the solutions are in close agreement. Away from the corners, even the 9 × 9 solutions are excellent. Due to the large gradients near the corners, the 9 × 9 mesh is too coarse and somewhat larger errors result.

Finally, the divergence form allows for the solution of the cavity flow at much larger Reynolds numbers. This has been demonstrated previously for the finite-difference calculations [15] and carries over to the spline development as well. Solutions have been obtained for Re = 1000. For Re = 1000, the solutions are moderately accurate and given in Table VIII and Fig. 7, where finite-difference results are also included [24]. The one to four correspondence is approximately satisfied; however,

TABLE VII

Comparison of Results for the Square Cavity, Re = 100

(a) Vorticity at the Center of the Moving Surface

Calculation method	Number of points	Moving surface center point vorticity
Spline 2	15 × 15	7.138
Spline 2	29 × 29	6.688
Finite difference	15 × 15	8.916
Finite difference	57 × 57	6.696
Extrapolated finite difference	—	6.548
Finite difference[a,b]	17 × 17	7.376
Finite difference[a,b]	65 × 65	6.609
Finite difference[a,b]	128 × 128	6.574
Spline 4[a]	17 × 17	6.532
Spline 4[a]	9 × 9	6.603
Hermite 4	15 × 15	6.927
Spline 4	17 × 17	6.610

(b) Maximum Stream Function

Calculation method	Number of points	Maximum stream function
Spline 2	15 × 15	−0.1053
Spline 2	29 × 29	−0.1043
Finite difference	15 × 15	−0.0874
Finite difference	57 × 57	−0.1013
Extrapolated finite difference	—	−0.1022
Finite difference[a,b]	17 × 17	−0.0987
Finite difference[a,b]	65 × 65	−0.1032
Finite difference[a,b]	128 × 128	−0.1034
Spline 4[a]	17 × 17	−0.1035
Spline 4[a]	9 × 9	−0.1072
Hermite 4	15 × 15	−0.1014
Spline 4	17 × 17	−0.1023

[a] Divergence form.
[b] Reference [24].

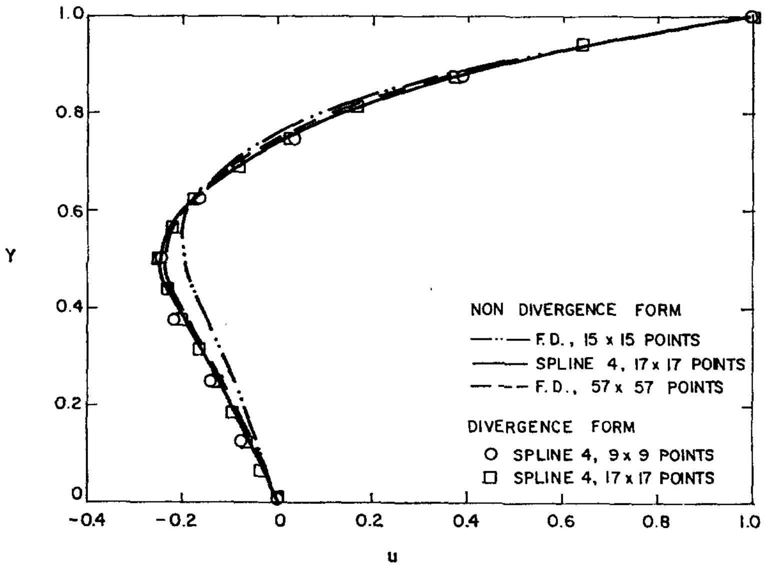


FIG. 6. Comparison of calculated velocity u through point of maximum ψ for $R_{\cdot} = 100$.

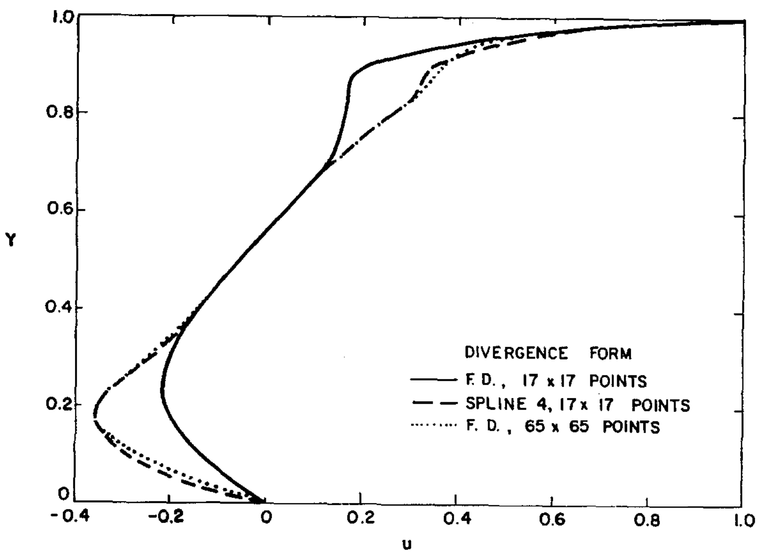


FIG. 7. Comparison of calculated velocity u through point of maximum ψ for Re = 1000.

TABLE VIII

Comparison of Results for the Square Cavity, Re = 1000

(a) Vorticity at the Center of the Moving Surface

Calculation method	Number of points	Moving surface center point vorticity
Spline 4[a]	17 × 17	14.254
Finite difference[a]	65 × 65	16.198
Finite difference[a]	17 × 17	21.508

(b) Maximum Stream Function

Calculation method	Number of points	Maximum stream function
Spline 4[a]	17 × 17	0.115
Finite difference[a]	65 × 65	0.114
Finite difference[a]	17 × 17	0.080

(c) Vorticity at the Vortex Center (x_c, y_c)

Calculation method	Number of points	Vorticity at Vortex Center (x_c, y_c)
Spline 4[a]	17 × 17	1.828 (0.56, 0.56)
Finite difference[a]	65 × 65	1.985 (0.52, 0.56)
Finite difference[a]	17 × 17	2.093 (0.56, 0.56)

[a] Divergence form.

the 17 × 17 grid is really too coarse for Re = 1000 and errors of about 10% are incurred. The stream function at the vortex center is within 1% of the finite-difference value (65 × 65). Finally, the vorticity values, 1.8–2.2, in the core region are quite close to the infinite Re limit of 1.886.

6. Summary

Polynomial spline interpolation has been used to develop a variety of higher-order collocation methods. Only those polynomials resulting in tridiagonal, or at worst 3 × 3 block-tridiagonal, matrix systems have been evaluated. The governing systems are obtained directly in terms of the functional values and certain derivatives of the functional values at the specified nodal points. The development is generally given for a nonuniform mesh, for which a high degree of accuracy is maintained.

Recently for a uniform mesh a so-called Padé, compact, Mehrstellung or Hermitian finite-difference procedure, which is 3×3 block-tridiagonal, has been proposed. It is shown that this formulation is a hybrid method resulting from two different polynomial splines. However, the Padé approximation is derived from a five-point discretization formula and might be difficult to extend to nonuniform mesh systems. The hybrid spline results apply to variable meshes. Also, the compact system of equations does not include certain simple relations between the first and second derivative approximations that are obtained from the polynomial spline interpolation formula. These relations are useful for reducing the size of the matrix system and thereby the computer time; in certain instances, boundary conditions can more easily be satisfied with these equations.

Finally, from three-point Taylor series expansions and Hermitian descritization of the functionals and their derivatives at the nodal points, an alternate derivation of the compact differencing scheme is presented. As only three nodal points are considered here, this procedure is less cumbersome than the Padé formulation and has been considered for nonuniform meshes and to develop systems with even higher-order truncation errors. Significantly, all of the polynomial spline block-tridiagonal systems can be recovered with this formulation. Moreover, a sixth-order (hybrid polynomial spline) 3×3 block-tridiagonal scheme has been devised. There does not appear to be any particular advantage of the polynomial spline formulation over the Hermitian discretization derivation. Polynomial interpolation can also be used to develop higher-order implicit temporal integration schemes, which have previously been developed by Hermite collocation, see Ref. [23].

The boundary conditions for all of the higher-order procedures have been obtained from the respective polynomial approximations. The truncation errors of all the procedures are presented in tabular form, and results are shown for a variety of viscous flow problems. Of the fourth-order methods, spline 4 [7, 8] has the smallest truncation error. From the solutions to the model problems, the increase in accuracy with decrease in truncation error is apparent. The sixth-order Hermite formulation leads to extraordinary accuracy even with very coarse grids.

An important conclusion of the present study is that, for equal accuracy, the spline 4 procedure requires one-fourth as many points, in a given direction, as a finite-difference calculation. This means less computer time and storage. Also, divergence form is preferable for coarse grids and/or large Reynolds numbers.

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